XXX. *Hints relating to the Uſe which may be made of the Tables of natural and logarithmic Sines, Tangents, &c. in the numerical Reſolution of adfected Equations. By* William Wales, *F. R. S. and Maſter of the Royal Mathematical School in* Chriſt's Hoſpital.

Read June 14, 1781.

THE firſt intimation that I can meet with relating to the uſe which may be made of the tables of ſines, tangents, and ſecants, in reſolving adfected equations, is in the latter part of the ſecond volume of Profeſſor SAUNDERSON's Elements of Algebra, printed in 1741, after his deceaſe. The profeſſor there ſhews how to reſolve thoſe two caſes which make the firſt and ſecond of the following examples, by means of the tables; but it appears, from many circumſtances, he was not aware that the third caſe could be reſolved in the ſame manner. All the three forms were, however, reſolved by the late Mr. ANTHONY THACKER, a very ingenious man, who died in the beginning of the year 1744, by the help of a ſet of tables, of his own invention; different from, but in ſome meaſure analogous to, the tables of ſines and tangents. Theſe tables were finiſhed and publiſhed, together with ſeveral papers concerning them, after his death, by a Mr. BROWN of Cleobury. In theſe papers, beſide explaining fully the uſe of the tables in reſolving cubic equations, Mr. THACKER ſhews that his method comprehends the reſolution of all biquadratic equations, if they be

first reduced to cubic ones in the manner which has been explained by DESCARTES and others, and the second term then taken away.

Since that time M. MAUDUIT has shewn how to find the roots of all the three forms of cubic equations, by means of the tables of sines, &c. in his excellent Treatise of Trigonometry. But none of these authors have attempted to resolve equations of more dimensions than three, by these means, without first reducing them to that number; nor even these, before the second term, or that which involves the square of the unknown quantity, is taken away: whereas such reductions will generally take up more time than is required to bring out the value of the unknown quantity by the following method; and, after all, frequently serve no other purpose but that of rendering the operation more intricate and troublesome.

The truly ingenious Mr. LANDEN, in his lucubrations, published in 1755, has given a general method of resolving that case of cubic equations, by means of the tables of sines, where all the roots are real, without the trouble of taking away the second term of the equation: and Mr. SIMPSON has shewn how to resolve equations of any dimensions, by the same means, provided those equations involve only the odd powers of the unknown quantity, and that the co-efficients observe such a law as will restrain the equation to that form which is expressive of the cosine of the multiple of an arc, of which the unknown quantity is the cosine, This was first done, I believe, by JOHN BERNOULLI, and afterwards by Mr. EULER, in his *Introduct. ad Analyt. Infinit.* and Mr. DE MOIVRE, in his *Miscell. Analyt.*; but the resolution of all equations of this form, as well as many others, is comprehended in the first of the following observations.

The firſt thought of extending the uſe of the tables of ſines, tangents, and ſecants, farther than to the caſes which have been already mentioned, occurred to me while I was conſidering the problem which produced the equation given in this paper as the fourth example. And it is remarkable, that the very ſame thought occurred to Dr. HUTTON about the ſame time, and in the reſolution of the ſame problem; and we were not a little ſurprized, on comparing our ſolutions together, to find that our ideas had taken ſo exactly the ſame turn; and that both ſhould have ſtumbled on a thought, which, as far as either of us knew, had never preſented itſelf to any one before. Having ſince examined farther into the matter, I have the ſatisfaction to find, that the principle is very extenſive, and that a great number of equations, eſpecially ſuch as ariſe in the practice of geometry, aſtronomy, and optics, may be reſolved by it with great eaſe and expedition.

But beſide the facility with which the value of the unknown quantity is brought out by means of the tables of ſines, tangents, and ſecants, this method of reſolution has another conſiderable advantage over moſt others which have been propoſed, inaſmuch as the firſt ſtate of the equation, without any previous reduction, is generally the beſt it can be in for reſolution; and from which it may moſt readily be diſcovered, how to ſeparate it into ſuch parts as expreſs the ſine, or the tangent, or the ſecant of the arc of a circle; or into the ſine, tangent, or ſecant of ſome multiple of that arc, or of a part of it; and in the doing of which conſiſts the principal part of the buſineſs in queſtion. It will alſo be of ſome advantage to preſerve the original ſubſtitutions as diſtinct as poſſible, by uſing only the ſigns of the ſeveral operations which it may be neceſſary to go

through

through in bringing the ſolution of a problem to an equation, inſtead of performing the operations themſelves.

Beſides the advantages which this method of procedure affords to the mode of reſolution now more particularly under conſideration, it has ſo many others over that which is commonly made uſe of, that I am much ſurprized the latter ſhould ever have been adopted. By preſerving thus the original ſubſtitutions diſtinct, all the way through an operation, every expreſſion, even to the final equation, will exhibit the whole proceſs up to that ſtep; and it will appear as clearly, how every expreſſion has been derived, as it does in that mode of analyſis which was uſed by the ancient geometricians: whereas, when the ſeveral original expreſſions are melted down into one maſs by the multitude of actual additions, ſubtractions, multiplications, and diviſions, which they generally undergo, in a long algebraical proceſs, conducted in the uſual manner, it is impoſſible to trace the ſmalleſt veſtige of the original quantities in the final equation, except ſuch as are repreſented by a ſingle letter. Of courſe, however obvious the ſeveral ſteps might be at the time when they were taken, every idea of them muſt be totally loſt in the reſult; and it will be utterly impoſſible to trace them back again, in the manner they are done in the compoſition of a problem, the ſolution of which has been inveſtigated by the geometrical analyſis *. Let me add, that it is to this cauſe

* This ſubject, if ever I am bleſſed with more leiſure than is at preſent my lot, ſhall be purſued farther in another paper: in which I ſhall endeavour to ſhew, that, notwithſtanding the great difference which there appears to be between algebra and geometry, they are really but one ſcience, differently treated; and that the operations of the former may be rendered as clear and perſpicuous as thoſe of the latter are allowed to be. A diſquiſition of this nature will at leaſt have the merit of reſcuing a very uſeful and expeditious mode of inveſtigation from

cause we must attribute all that obscurity which the algebraic mode of investigation has been so frequently charged with.

I shall endeavour to verify this doctrine, in some measure, by the expressions which are put down in the following tables for the sines, cosines, and tangents of arcs of circles, and of the multiples of those arcs; which tables will be found very useful in the prosecution of the design which I am now upon, and are absolutely necessary in the explanation of it.

from an unmerited stigma: and if I never be happy enough to have an opportunity of doing it myself, what I have here said may be the means of putting some other person, who has, upon it.

TABLE

T A B L

Arc.	Sine.	Cosine.
A	x	$\sqrt{\overline{r+x}\cdot\overline{r-x}}$
2A	$\frac{2x}{r}\cdot\sqrt{\overline{r+x}\cdot\overline{r-x}}$	$\frac{\overline{r+x}\cdot\overline{r-x}-x^2}{r}$
3A	$\frac{3x\cdot\overline{r+x}\cdot\overline{r-x}-x^3}{r^2}$	$\frac{\overline{r+x}\cdot\overline{r-x}-3x^2}{r^2}\sqrt{\overline{r+x}\cdot\overline{r-}}$
4A	$\frac{4x\cdot\overline{r+x}\cdot\overline{r-x}-4x^3}{r^3}\sqrt{\overline{r+x}\cdot\overline{r-x}}$	$\frac{\overline{r+x}^2\cdot\overline{r-x}^2-6x^2\cdot\overline{r+x}\cdot\overline{r-}}{r^3}$
5A	$\frac{5x\cdot\overline{r+x}^2\cdot\overline{r-x}^2-10x^3\cdot\overline{r+x}\cdot\overline{r-x}+x^5}{r^4}$	$\frac{\overline{r+x}^2\cdot\overline{r-x}^2-10x^2\cdot\overline{r+x}\cdot\overline{r-x}+5x^4}{r^4}$
6A	$\frac{6x\cdot\overline{r+x}^2\cdot\overline{r-x}^2-20x^3\cdot\overline{r+x}\cdot\overline{r-x}+6x^5}{r^5}\sqrt{\overline{r+x}\cdot\overline{r-x}}$	$\frac{\overline{r+x}^3\cdot\overline{r-x}^3-15x^2\cdot\overline{r+x}^2\cdot\overline{r-x}^2+15x}{r^5}$

T A B L

Arc.	Sine.	Cosine.
A	$\frac{rx}{\sqrt{r^2+x^2}}$	$r\cdot\frac{r}{\sqrt{r^2+x^2}}$
2A	$\frac{2r^2x}{r^2+x^2}$	$r\cdot\frac{\overline{r+x}\cdot\overline{r-x}}{r^2+x^2}$
3A	$\frac{\overline{r+x}\cdot\overline{r-x}+2r^2}{r^2+x^2}\times\frac{rx}{\sqrt{r^2+x^2}}$	$\frac{r\cdot\overline{r+x}\cdot\overline{r-x}-2rx}{r^2+x^2}\times\frac{}{\sqrt{r^2}}$
4A	$\frac{2\cdot\overline{r+x}\cdot\overline{r-x}}{r^2+x^2}\times\frac{2r^2x}{r^2+x^2}$	$r\cdot\frac{\overline{r+x}\cdot\overline{r-x}-2rx}{r^2+x^2}\times\frac{\overline{r+x}\cdot\overline{r-}}{r^2+}$
5A	$\frac{\overline{r+x}^2\cdot\overline{r-x}^2+4r^2\cdot\overline{r+x}\cdot\overline{r-x}-4r^2x^2}{\overline{r^2+x^2}\cdot\overline{r^2+x^2}}\times\frac{rx}{\sqrt{r^2+x^2}}$	$r\cdot\frac{\overline{r+x}^2\cdot\overline{r-x}^2-4x^2\cdot\overline{r+x}\cdot\overline{r-x}-4}{\overline{r^2+x^2}\cdot\overline{r^2+x^2}}$
6A	$\frac{3\cdot\overline{r+x}^2\cdot\overline{r-x}^2-4r^2x^2}{\overline{r^2+x^2}\cdot\overline{r^2+x^2}}\times\frac{2\cdot r^2x}{r^2+x^2}$	$\frac{\overline{r+x}^2\cdot\overline{r-x}^2-12r^2x^2}{r^2+x^2}\times\frac{\overline{r+x}\cdot\overline{r-}}{r^2+x^2}$

BLE I.

Cosine.	Tangent.
$\overline{r+x}\cdot\overline{r-x}$	$\frac{rx}{\sqrt{\overline{r+x}\cdot\overline{r-x}}}$
$\frac{\ldots x\cdot\overline{r-x}-x^2}{r}$	$\frac{2}{\overline{r+x}\cdot\overline{r-x}-x^2}\sqrt{\frac{rx}{\overline{r+x}\cdot\overline{r-x}}}$
$-3x^2\sqrt{\overline{r+x}\cdot\overline{r-x}}$	$\frac{3\cdot\overline{r+x}\cdot\overline{r-x}-x^2}{\overline{r+x}\cdot\overline{r-x}-3x^2}\times\frac{rx}{\sqrt{\overline{r+x}\cdot\overline{r-x}}}$
$\frac{-6x^2\cdot\overline{r+x}\cdot\overline{r-x}+x^4}{r^3}$	$\frac{4\cdot\overline{\overline{r+x}\cdot\overline{r-x}-4x}^2}{\overline{r+x}\vert^2\cdot\overline{r-x}\vert^2-6x^2\cdot\overline{r+x}\cdot\overline{r-x}+x^4}\sqrt{\frac{rx}{\overline{r+x}\cdot\overline{r-x}}}$
$\overline{r+x}\cdot\overline{r-x}+5x^4\sqrt{\overline{r+x}\cdot\overline{r-x}}$	$\frac{5\cdot\overline{r+x}\vert^2\cdot\overline{r-x}\vert^2-10x^2\cdot\overline{r+x}\cdot\overline{r-x}+x^4}{\overline{r+x}\vert^2\cdot\overline{r-x}\vert^2-10x^2\cdot\overline{r+x}\cdot\overline{r-x}+5x^4}\times\frac{rx}{\sqrt{\overline{r+x}\cdot\overline{r-x}}}$
$\frac{\ldots\vert^2\cdot\overline{r-x}\vert^2+15x^4\cdot\overline{r+x}\cdot\overline{r-x}-x^6}{r^5}$	$\frac{6\cdot\overline{r+x}\vert^2\cdot\overline{r-x}\vert^2-20x^2\cdot\overline{r+x}\cdot\overline{r-x}+6x^4 \quad rx}{\overline{r+x}\vert^3\cdot\overline{r-x}\vert^3-15x^2\cdot\overline{r+x}\vert^2\cdot\overline{r-x}\vert^2+15x^4\cdot\overline{r+x}\cdot\overline{r-x}\cdot\sqrt{\overline{r+x}\cdot\overline{r-x}}}$

BLE II.

Cosine.	Tangent.
$\frac{r}{\sqrt{r^2+x^2}}$	x
$\frac{\overline{r+x}\cdot\overline{r-x}}{r^2+x^2}$	$\frac{2r^2x}{\overline{r+x}\cdot\overline{r-x}}$
$\frac{\ldots-x-2rx}{\ldots x^2}\times\frac{r}{\sqrt{r^2+x^2}}$	$\frac{x\cdot\overline{r+x}\cdot\overline{r-x}+2r^2x}{\overline{r+x}\cdot\overline{r-x}-2x^2}$
$\frac{2rx}{\ldots}\times\frac{\overline{r+x}\cdot\overline{r-x}+2rx}{r^2+x^2}$	$\frac{4r^2x\cdot\overline{r+x}\cdot\overline{r-x}}{\overline{r+x}\vert^2\cdot\overline{r-x}\vert^2-4r^2x^2}$
$\frac{\ldots\cdot\overline{r+x}\cdot\overline{r-x}-4r^2x^2}{r^2+x^2}\times\frac{r}{\sqrt{r^2+x^2}}$	$\frac{x\cdot\overline{r+x}\vert^2\cdot\overline{r-x}\vert^2+4r^2x\cdot\overline{r+x}\cdot\overline{r-x}-4r^2x^3}{\overline{r+x}\vert^2\cdot\overline{r-x}\vert^2-4x^2\cdot\overline{r+x}\cdot\overline{r-x}-4r^2x^2}$
$\ldots x^2\times\frac{\overline{r+x}\cdot\overline{r-x}}{r^2+x^2}\times\frac{r}{r^2+x^2}$	$r^2\cdot\frac{6x\cdot\overline{r+x}\vert^2\cdot\overline{r-x}\vert^2-8r^2x^3}{\overline{r+x}\vert^3\cdot\overline{r-x}\vert^3-12r^2x^2\cdot\overline{r+x}\cdot\overline{r-x}}$

TABLE III

Arc.	Sine.	Cosine.
A	$\frac{r\cdot\sqrt{\overline{x+r}\cdot\overline{x-r}}}{x}$	$r\cdot\frac{r}{x}$
2A	$r\cdot\frac{2r\cdot\sqrt{\overline{x+r}\cdot\overline{x-r}}}{x^2}$	$r\cdot\frac{r^2-\overline{x+r}\cdot\overline{x-r}}{x^2}$
3A	$r\cdot\frac{3r^2-\overline{x+r}\cdot\overline{x-r}}{x^3}\sqrt{\overline{x+r}\cdot\overline{x-r}}$	$r\cdot\frac{r^3-3r\cdot\overline{x+r}\cdot\overline{x-r}}{x^3}$
4A	$\frac{r\cdot 4r^3-4r\cdot\overline{x+r}\cdot\overline{x-r}}{x^4}\sqrt{\overline{x+r}\cdot\overline{x-r}}$	$r\cdot\frac{r^4-6r^2\cdot\overline{x+r}\cdot\overline{x-r}+\overline{x+r}^{\,2}\cdot\overline{x-r}^{\,2}}{x^4}$
5A	$r\cdot\frac{5r^4-20r^2\cdot\overline{x+r}\cdot\overline{x-r}+\overline{x+r}^{\,2}\cdot\overline{x-r}^{\,2}}{x^5}\sqrt{\overline{x+r}\cdot\overline{x-r}}$	$r\cdot\frac{r^5-10r^3\cdot\overline{x+r}\cdot\overline{x-r}+5r\cdot\overline{x+r}^{\,2}\cdot\overline{x-r}^{\,2}}{x^5}$
6A	$r\cdot\frac{6r^5-20r^3\cdot\overline{r+x}\cdot\overline{r-x}+6r\cdot\overline{x+r}^{\,2}\cdot\overline{x-r}^{\,2}}{x^6}\sqrt{\overline{x+r}\cdot\overline{x-r}}$	$r\cdot\frac{r^6-15r^4\cdot\overline{x+r}\cdot\overline{x-r}+15r^2\cdot\overline{x+r}^{\,2}\cdot\overline{x-r}^{\,2}-\overline{x+r}^{\,3}\cdot\overline{x-}}{x^6}$

TABLE IV.

Arc.	Sine.	Cosine.
A	$\sqrt{r^2-\overline{r-x}^{\,2}}$	$r-x$
2A	$2\cdot\frac{r-x}{r}\sqrt{r^2-\overline{r-x}^{\,2}}$	$\frac{2\cdot\overline{r-x}^{\,2}-r^2}{r}$
3A	$\frac{4\cdot\overline{r-x}^{\,2}-r^2}{r^2}\sqrt{r^2-\overline{r-x}^{\,2}}$	$\frac{4\cdot\overline{r-x}^{\,3}-3r^2\cdot\overline{r-x}}{r^2}$
4A	$\frac{8\cdot\overline{r-x}^{\,3}-4r^2\cdot\overline{r-x}}{r^3}\sqrt{r^2-\overline{r-x}^{\,2}}$	$\frac{8\cdot\overline{r-x}^{\,4}-8r^2\cdot\overline{r-x}^{\,2}+r^4}{r^3}$
5A	$\frac{16\cdot\overline{r-x}^{\,4}-12r^2\cdot\overline{r-x}^{\,2}+r^4}{x^4}\sqrt{r^2-\overline{r-x}^{\,2}}$	$\frac{16\cdot\overline{r-x}^{\,5}-20r^2\cdot\overline{r-x}^{\,3}+5r^4\cdot\overline{r-x}}{r^4}$
6A	$\frac{32\cdot\overline{r-x}^{\,5}-32r^2\cdot\overline{r-x}^{\,3}+6r^4\cdot\overline{r-x}}{r^5}\sqrt{r^2-\overline{r-x}^{\,2}}$	$\frac{32\cdot\overline{r-x}^{\,6}-48r^2\cdot\overline{r-x}^{\,4}+18r^4\cdot\overline{r-x}^{\,2}-r^6}{r^5}$

T A B L E III.

Cosine.	Tangent.
$r \cdot \frac{r}{x}$	$\sqrt{\overline{x+r}\cdot\overline{x-r}}$
$r \cdot \frac{r^2-\overline{x+r}\cdot\overline{x-r}}{x^2}$	$\frac{2r^2\cdot\sqrt{\overline{x+r}\cdot\overline{x-r}}}{r^2-\overline{x+r}\cdot\overline{x-r}}$
$r \cdot \frac{r^3-3r\cdot\overline{x+r}\cdot\overline{x-r}}{x^3}$	$\frac{3r^2-\overline{x+r}\cdot\overline{x-r}}{r-3\cdot\overline{x+r}\cdot\overline{x-r}}\sqrt{\overline{x+r}\cdot\overline{x-r}}$
$\cdot \frac{r^4-6r^2\cdot\overline{x+r}\cdot\overline{x-r}+\overline{x+r}^{\,2}\cdot\overline{x-r}^{\,2}}{x^4}$	$\frac{4r^4-4r^2\cdot\overline{x+r}\cdot\overline{x-r}}{r^4-6r^2\cdot\overline{x+r}\cdot\overline{x-r}+\overline{x+r}^{\,2}\cdot\overline{x-r}^{\,2}}\sqrt{\overline{x+r}\cdot\overline{x-r}}$
$\frac{-10r^3\cdot\overline{x+r}\cdot\overline{x-r}+5r\cdot\overline{x+r}^{\,2}\cdot\overline{x-r}^{\,2}}{x^5}$	$\frac{5r^4-10r^2\cdot\overline{x+r}\cdot\overline{x-r}+\overline{x+r}^{\,2}\cdot\overline{x-r}^{\,2}}{r^4-10r^2\cdot\overline{x+r}\cdot\overline{x-r}+5\cdot\overline{x+r}^{\,2}\cdot\overline{x-r}^{\,2}}\sqrt{\overline{x+r}\cdot\overline{x-r}}$
$\frac{\overline{x+r}\cdot\overline{x-r}+15r^2\cdot\overline{x+r}^{\,2}\cdot\overline{x-r}^{\,2}-\overline{x+r}^{\,3}\cdot\overline{x-r}^{\,3}}{x^6}$	$\frac{6r^6-20r^4\cdot\overline{x+r}\cdot\overline{x-r}+6r^2\cdot\overline{x+r}^{\,2}\cdot\overline{x-r}^{\,2}}{r^6-15r^4\cdot\overline{x+r}\cdot\overline{x-r}+15r^2\cdot\overline{x+r}^{\,2}\cdot\overline{x-r}^{\,2}-\overline{x+r}^{\,3}\cdot\overline{x-r}^{\,3}}\sqrt{\overline{x+r}\cdot\overline{x-r}}$

A B L E IV.

Cosine.	Tangent.
$r-x$	$\frac{r}{r-x}\sqrt{r^2-\overline{r-x}^{\,2}}$
$\frac{2\cdot\overline{r-x}^{\,2}-r^2}{r}$	$\frac{2r\cdot\overline{r-x}}{2\cdot\overline{r-x}^{\,2}-r^2}\sqrt{r^2-\overline{r-x}^{\,2}}$
$\frac{4\cdot\overline{r-x}^{\,3}-3r^2\cdot\overline{r-x}}{r^2}$	$\frac{4r\cdot\overline{r-x}^{\,2}-r^3}{4\cdot\overline{r-x}^{\,3}-3r^2\cdot\overline{r-x}}\sqrt{r^2-\overline{r-x}^{\,2}}$
$\frac{8\cdot\overline{r-x}^{\,4}-8r^2\cdot\overline{r-x}^{\,2}+r^4}{r^3}$	$\frac{8r\cdot\overline{r-x}^{\,3}-4r^3\cdot\overline{r-x}}{8\cdot\overline{r-x}^{\,4}-8r^2\cdot\overline{r-x}^{\,2}+r^4}\sqrt{r^2-\overline{r-x}^{\,2}}$
$\frac{16\cdot\overline{r-x}^{\,5}-20r^2\cdot\overline{r-x}^{\,3}+5r^4\cdot\overline{r-x}}{r^4}$	$\frac{16r\cdot\overline{r-x}^{\,4}-12r^3\cdot\overline{r-x}^{\,2}+r^5}{16\cdot\overline{r-x}^{\,5}-20r^2\cdot\overline{r-x}^{\,3}+5r^4\cdot\overline{r-x}}\sqrt{r^2-\overline{r-x}^{\,2}}$
$\frac{\cdot\overline{r-x}^{\,6}-48r^2\cdot\overline{r-x}^{\,4}+18r^4\cdot\overline{r-x}^{\,2}-r^6}{r^5}$	$\frac{32r\cdot\overline{r-x}^{\,5}-32r^3\cdot\overline{r-x}^{\,3}+6r^5\cdot\overline{r-x}}{32\cdot\overline{r+x}^{\,6}-48r^2\cdot\overline{r-x}^{\,4}+18r^4\cdot\overline{r+x}^{\,2}-r^6}\sqrt{r^2-\overline{r-x}^{\,2}}$

Observa

Observations on the foregoing tables.

EACH of the formulæ in these tables may be considered as one side of an equation, involving the unknown quantity x to different dimensions. In some of the formulæ the odd powers of x are only found, in others the even ones alone, and in others both; but they are all equally useful in finding the value of the unknown quantity in adfected equations which contain all the powers of that quantity, as will plainly appear from the following considerations.

I. If, on bringing the solution of any problem to an equation with some known quantity, it be found to correspond with any of the formula in these tables; or, if by any means it can be reduced to any of them, it is manifest, that nothing remains to be done but to divide the known side of the equation by the value of the quantity which is here denoted by r, and to seek for the quotient in the tables of sines, cosines, or tangents, as the case may require, and the value of the unknown quantity will be the sine, tangent, secant, or versed sine, of a given part of that arc (according as the expression is found in the first, second, third, or fourth table) multiplied by the value of r.

II. If, as will more frequently happen, the final equation of an operation be found equivalent to the sum, difference, product, or quotient, of some two or more of these formula; or to the sum, difference, product, or quotient, of some two or more of them multiplied or divided, increased or lessened, by some known quantity or quantities; then, having taken away

the known quantities by the common algebraic rules, obferve the following ones.

1ft. When the equation is found to correfpond with the fum or difference of two formula in thefe tables, which are the fine and tangent, fine and cofine, or cofine and tangent, of the fame arc, by running the eye along the tables of natural fines and tangents, find thefe two arcs, immediately following one another, the fum or difference of the fine and tangent, fine and cofine, or cofine and tangent, of which are one of them greater, and the other lefs than the number which conftitutes the known fide of the equation. Take the excefs of one of thefe fums or differences above, and what the other fum or difference wants of the faid given number, add thefe two errors together, and fay, as the fum of them is to 60″, fo is that error which belongs to the lefs arc to a number of feconds; which being added to the lefs arc will give one, the fum or difference of whofe fine and tangent, fine and cofine, or cofine and tangent, is exactly equal to the number which conftitutes the known fide of the equation. Of the arc, thus found, let fuch a part be taken as the table in which the formula are found directs, and the natural fine, tangent, fecant, or verfed fine (as the cafe may require) of this part, being multiplied by the value of r, if r be found in the equation, will be the value of x fought.

2d. When the equation happens to be the product or quotient of two formulæ which exprefs the fine and cofine, fine and tangent, or cofine and tangent, of the fame arc, take the logarithm of the number which conftitutes the known fide of the equation, and then follow exactly the directions given in the firft cafe, ufing the tables of logarithmic fines and tangents inftead of the tables of natural ones.

3d. If the equation, finally resulting from the resolution of any problem, present itself in an expression which is composed of the sum or difference of the sine, cosine, or tangent, of an arc, of which the unknown quantity is the sine, cosine, tangent, or versed sine, and the sine, cosine, or tangent, of some multiple of that arc, it will then be convenient to have two tables of sines and tangents; and in running the eye along them to find the two arcs immediately following one another, of which the sum or difference of the sine, cosine, or tangent, of one of them, and the sine, cosine, or tangent, of some multiple of it, may be less, and the sum or difference of the sine, cosine, or tangent, of the other, and the sine, cosine, or tangent, of the same multiple of it, may be greater than the number which constitutes the known side of the equation, for every minute of a degree that the finger is moved over in one, it must be moved over a number of minutes in the other, which is equal to the number of times that the single arc is contained in the multiple one. When these two arcs are found, the operation will not differ so materially from that which is pointed out in the first rule as to merit repetition.

4th. If, instead of the sum or difference of the sine, cosine, or tangent, of an arc, and the sine, cosine, or tangent, of some multiple of it, the form of the equation be such as to be constituted of the product of them, or the quotient of one divided by the other, the last rule will still hold good, using only the logarithmic sines and tangents instead of the natural ones, and comparing the sum or difference of them, according as the equation is composed of the product or quotient of the two factors, with the logarithm of the number which constitutes the known side of the equation, instead of that number itself.

5th. Sometimes the final equation will come out in expressions which are constituted of the sum, difference, product, or quotient, of the sine, cosine, or tangent, of some multiple of an arc, of which the unknown quantity is the sine, tangent, secant, or versed sine, and the sine, cosine, or tangent, of some other multiple of the same arc. And in any of these cases it is manifest, that the method of proceeding, in order to obtain one of the multiple arcs, and from thence the single one, of which the unknown quantity is the sine, tangent, &c. will not be greatly different from those which have been described in the third and fourth rules. The most material difference consists in this, that instead of proceeding minute by minute, according to the directions in the third rule to find the single arc, it will now be most convenient to proceed in each table by as many minutes at each step as are equal in number to the number of times which the single arc is contained in the multiple ones respectively.

6th. Equations will frequently make their appearance in formulæ which express the square, cube, &c. of the sine, cosine, or tangent, of the multiple of some arc, of which the unknown quantity is the sine, tangent, secant, or versed sine; or in formulæ which are expressive of the sum, difference, product, &c. of the sine, cosine, or tangent, of an arc, and some power of the sine, cosine, or tangent, of the same arc; or of some multiple of it, the unknown quantity being some other trigonometrical line belonging to that arc. Or the equation may be compounded of the sum, difference, product, &c. of the same, or different powers of the sines, tangents, or cosines, of different multiples of an arc, the unknown quantity being the sine, tangent, secant, or versed sine, of that arc. In every one of these cases the tables will give the value of the unknown

quantity,

quantity, and in moſt of them with great eaſe and expedition. The method which is to be purſued in each caſe will readily preſent itſelf to a ſkilful analyſt, who attends carefully to what has been already ſaid, and to the examples which follow.

IV. The formula in the four preceding tables may be greatly varied by ſuppoſing x, the unknown quantity, to be ſome part or parts of the ſine, tangent, &c. as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{3}{4}$, &c. or ſome multiple of it, as twice, thrice, &c. Or x may be the ſquare, or the ſquare root, or any other power of the ſine, tangent, ſecant, or verſed ſine, of an arc; in every one of which caſes the formula will put on different appearances, either with reſpect to the powers or co-efficients of the unknown quantity, and yet admit of the ſame kind of application.

V. The tables may be rendered yet more extenſively uſeful by inſerting expreſſions for the ſines, coſines, and tangents, of half the arc which has x for its ſine, tangent, ſecant, or verſed ſine; and alſo for the ſines, coſines, and tangents, of the odd multiples of this half arc, which expreſſions, together with thoſe already inſerted, may be conſidered as the ſines, coſines, and tangents, of the multiples of an arc, the unknown quantity, being the ſine, tangent, &c. of twice that arc. And this conſideration may ſometimes be applied to very uſeful purpoſes.

VI. In order to render the formula in the tables more general, I have put r for the radius of the circle; whereas it will frequently happen, that the equation, finally reſulting from the reſolution of a problem, eſpecially thoſe which relate to the doctrine of the ſphere, will preſent itſelf in a form where the radius muſt be taken equal to unity: what theſe forms are will readily appear by ſubſtituting unity for r and its powers every where in the expreſſion.

It would be endleſs were I to undertake to enumerate all the various circumſtances and caſes in which this method of bringing out the unknown quantity may be applied with ſucceſs: what has already been ſaid will be ſufficient to explain the nature of it, and to enable the analyſt to apply it in other inſtances as they occur to him, I ſhall therefore only add a few examples to illuſtrate it.

EXAMPLE I.

Let it be required to find the value of x in an equation of the form $x^3 - r^2 x = a$.

If r^2 be expounded by 50, and a by 120 (ſee Phil. Tranſ. vol. LXVIII. p. 937.) the equation may be reduced to $\sqrt{x} \times \sqrt{x^2 - 50} = \sqrt{120}$; and, conſequently, by tab. III. if x be conſidered as the ſecant of an arc, of which the radius is $\sqrt{50}$, $\sqrt{x^2 - 50}$ will be the tangent of it, and we ſhall have to find an arc, ſuch that the tangent multiplied by the ſquare root of the ſecant may be equal to $\sqrt{120}$; or, which amounts to the ſame thing, ſuch an arc that the log. tang. together with half the log. ſecant may be equal to half the log. of 120. But becauſe the tangent and ſecant, here required, are to the radius of the $\sqrt{50}$, the log. tangents and ſecants in the tables muſt be increaſed by the logarithm of that number, and therefore log. tang. $+ \frac{1}{2}$ log. 50 $+ \frac{1}{2}$ log. ſecant $+ \frac{1}{4}$ log. 50 $= \frac{1}{2}$ log. 120: or log. tang. $+ \frac{1}{2}$ log. ſecant $= \frac{1}{2}$ log. 120 $- \frac{3}{4}$ log. of 50. Hence, having taken $\frac{3}{4}$ the log. of 50 from $\frac{1}{2}$ the log. of 120, run the eye along the tables of logarithmic tangents and ſecants until an arc be found of which the ſum of the log. tangent and half the log. ſecant is equal to 19.7653631, the remainder.

In

In this manner it will be readily found, that the ſum of the log. tangent and half the log. ſecant of 28° 37′ is leſs than that difference by 2012, and that the ſum of the log. tangent and half the log. ſecant of 28° 38′ is greater than it by 1337: therefore 3349 (2012 + 1337) : 60″ :: 2012 : 36″. The exact arc, therefore, of which the ſum of the log. tangent and half the log. ſecant is equal to 19.7653631 is 28° 37′ 36″, and the log. ſecant of it is 10.0566242, which being increaſed by 0.8494850, the log. of $\sqrt{50}$ gives 0.9061092, which is the logarithm of 8.055810, the value of x ſought, and which is true to ſeven places of figures.

EXAMPLE II.

To find the value of x in an equation of the form $x^3 - r^2x = -a$.

If r be expounded by 3, and a by 10, as they are in the example, given at p. 433. of the Phil. Tranſ. vol. LXX. the equation will be $x^3 - 9x = -10$, and may be transformed to $\sqrt{x} \times \sqrt{9 - x^2} = \sqrt{10}$; and, therefore, by tab. I. the ſquare root of the ſine into the coſine of an arc, of which the radius is 3, is equal to the ſquare root of 10. Conſequently, an arc muſt be found, ſuch that the ſum of the log. coſine and half the log. tangent is equal to half the log. of 10. But becauſe the radius of this arc muſt be 3, the log. ſines and coſines muſt be increaſed by the log. of 3; and, therefore, log. coſ. + log. of 3 + $\frac{1}{2}$ log. ſine + $\frac{1}{2}$ log. of 3 muſt be equal to half the log. of 10; or, an arc muſt be found of which the ſum of the tabular log. coſine and half the log. ſine is equal to the difference between half the log. of 10 and 1$\frac{1}{2}$ the log. of 3. Hence, having ſubtracted 1$\frac{1}{2}$ log. of 3 from half the log. of 10, run the eye

along GARDINER's tables of logarithmic sines, by which means it will be readily found, that the sum of the log. cosine and half the log. sine of 28° 53′ 30″ is less than 19.7843181, the excess of half the log. of 10 above $1\frac{1}{2}$ log. 3, by 15, and that the sum of the log. cosine and half the log. sine of 28° 53′ 40″ is greater than that difference by 60. Consequently 75 (15 + 60) : 10″ :: 15 : 2″. The exact arc, therefore, of which the sum of the log. cosine and half the log. sine is equal to 19.7843181, is 28° 53′ 32″; and the log. sine of this arc, increased by the log. of 3, is 0.1612153, the logarithm of 1.44949, the value of x required, true to the last place.

But many equations of this form, and this example among the rest, admit of two positive values of the unknown quantity; and by carrying the eye farther along the tables it will be found also, that the sum of the log. cosine and half the log. sine of 41° 48′ 30″ is greater than 19.7843181 by 50, and that the sum of the log. cosine and half the log. sine of 41° 48′ 40″ is too little by 21. Consequently, 71 (50 + 21) : 10″ :: 50 : 7″: of course, 41° 48′ 37″ is another arc, of which the sum of the log. cosine and half the log. sine is equal to 19.7843181, and the log. sine of this arc, increased by the log. of 3, is the logarithm of 1.999999, another value of x, and which errs but by unity in the seventh place.

The third root, as it is generally called, of this equation, which is necessarily negative, and equal to the sum of the other two, belongs properly to the equation which is given as the first example, of which it is the affirmative root, and may be found by the directions which are there given.

EXAMPLE

EXAMPLE III.

To find the value of x in an equation of the form $x^3 + r^2x = a$.

Let us take as examples of this equation $x^3 + 3x = .04$, $x^3 + 3x = .08$, and $x^3 + 3x = .12$, which are three of the inftances given by Dr. HALLEY, in his Synopfis of the Aftronomy of Comets, to illuftrate the mode of computation that he purfued in conftructing his general table for calculating the place of a comet in a parabolic orbit: and it is obvious, a being put for the known fide of the equation, that it may be transformed to $\sqrt{x} \times \sqrt{3 + x^2} = \sqrt{a}$: where, if x be confidered as the tangent of an arc, the radius of which is $\sqrt{3}$, $\sqrt{3 + x^2}$ will be the fecant of that arc; and, confequently, by what is fhewn in the firft example, an arc muft be found, fuch, that the fum of the tabular log. fecant and half the tabular log. tangent may be equal to the excefs of half the log. of a above $\frac{3}{4}$ of the log. of 3. In the firft of the above three inftances this excefs will be found, 18.9431891, in the fecond 19.0937041, and in the third 19.1817497; and by running the eye along GARDINER's Tables of Logarithmic Sines and Tangents, it will be found, that the firft falls between 0° 26′ 20″ and 0° 26′ 30″, the fecond between 0° 52′ 50″ and 0° 53′ 0″, and the third between 1° 19′ 20″ and 1° 19′ 30″; and, by purfuing the mode which has been defcribed in the two former examples, the exact arcs will be found 0° 26′ 27″,7, 0° 54′ 51″,7, and 1° 19′ 20″,1, and their refpective tangents, to the radius $\sqrt{3}$, .01333248, .0266611, and .0399787, the three values of x fought. And in this manner Dr. HALLEY's table may be extended to any length with the utmoft eafe, expedition, and accuracy.

3 Thus

Thus far this matter has been carried by former writers; but those who may be at the trouble of consulting them will find that I have not copied their methods: on the contrary, these which are given here are more plain and obvious than theirs are, and the operations considerably shorter. What follows has not, I believe, been adverted to by any before me.

EXAMPLE IV.

Let the equation arising from the proportion $a : b+x \cdot \overline{1-c^2} :: c\sqrt{1-x^2} : c^2x$ be taken, which is the result of an inquiry into the situation of that place on the surface of the earth, considered as a spheroid, which is at the greatest distance from a given one, suppose London. In this inquiry a and b were put to represent the sine and cosine of the latitude of the given place (in the spheroid); c for $\frac{229}{230}$, the ratio of the axes; and x for the sine of the distance of the required place from the opposite pole (in the spheroid also). The equation, which is of four dimensions with all the terms, is manifestly $acx = \overline{b+x \cdot \overline{1-c^2}} \times \sqrt{1-x^2}$, or $\frac{x}{\sqrt{1-x^2}} - x \cdot \frac{1-c^2}{ac} = \frac{b}{ac}$; in which it is evident from tab. I. that the difference between the tangent and the product of the sine into a given quantity is known. In order, therefore, to find the value of x, compute $\frac{b}{ac}$, and $\frac{1-c^2}{ac}$, and find the logarithm of the latter. Now, because the elliptic meridian differs but little from a circle, the place sought will not be far from the antipodes of the given one, and its distance from the opposite pole may therefore be estimated at

$39° 5'$;

$39^\circ\ 5'$; and, having taken out the natural tangent, and logarithmic fine of this arc, add the logarithm of $\frac{1-c^2}{ac}$ to the latter, and find the number correfponding to the fum, which will be lefs than the natural tangent of $39^\circ\ 5'$ by 2869. As this affumption is fo near, take $39'\ 6''$ for the next, repeat the operation, and the refult will be 1935 too great. Then 4804 $(2869+1935) : 60'' :: 2869 : 36''$; which being added to $39^\circ 5'$, gives $39^\circ\ 5'\ 36''$, for the co-latitude of the place fought, and the natural fine of this arc, or .6305856 is the value of x in this equation.

EXAMPLE V.

Let the equation $x^3 + \frac{b^2-2a^2}{4a}x^2 + \frac{2a^2-b^2}{2}x - \frac{ab^2}{4} = 0$, be taken, which refults from a folution of one cafe of the problem *de inclinationibus* of APOLLONIUS; but which, as it naturally rifes to a folid problem, was not, I conceive, confidered by that celebrated author. The refult of the analyfis, before any reduction takes place, is this proportion, $x+a : x-a :: 2\sqrt{ax} : b$; and hence, $\frac{x-a}{x+a}\sqrt{ax} = \frac{1}{2}b$. But it is here manifeft, that if a be taken for the tangent of an arc, of which the radius is $\sqrt{ax}$, x will be the cotangent of it, and $\frac{x-a}{x+a}\sqrt{ax}(=\frac{1}{2}b)$ the cofine of twice that arc. Confequently, we have to find an arc, the tangent of which is to the cofine of twice that arc as a is to $\frac{1}{2}b$; and this being done, the natural co-tangent of that arc, to the proper radius, will be the value of x.

 Thus,

Thus, let a be 10, and b 24; and the difference of the logarithms of a and $\frac{1}{2}b$ will be 0.0791812. Now, by running the eye along GARDINER's Tables of logarithmic Sines and Tangents, it will be readily seen, that the log. tangent of 26° 33′ 50″, when increased by 0.0791812, is less than the cosine of 53° 7′ 40″ twice that arc; and that the log. tangent of 26° 34′ 0″, when increased by the same quantity, is too great. And, by actually taking out the logarithms, and making the additions, the former will be found too small by 455, and the latter too great by 632. Then, 1087 (455 + 632) : 10″ :: 455 : 4″; which being added to 26° 33′ 50″ gives 26° 33′ 54″ for the arc of which x is the co-tangent. And if to twice the log. co-tangent of this arc the logarithm of a (10) be added, the sum (1.6020600) is the log. of 40, the value of x sought.

EXAMPLE VI.

The equation resulting from a solution of the famous problem of ALHAZEN may be given as another example of the use of this method. Many solutions of this celebrated problem, by HUYGENS, SLUSIUS, and others, may be met with in the Philosophical Transactions. Solutions to it may also be found at the end of Dr. ROBERT SIMPSON's Conic Sections, in Dr. SMITH's Optics, Mr. ROBIN's Mathematical Tracts, and other places; but the most direct and obvious method is, perhaps, that which follows.

Put $a = DC$, $b = dC$, $r = CI$; $x = CB$ and $y = CE$, the cosines of the arcs IA, IH, to the radius r: then will the sines of those arcs, BA, EH, be expressed by $\sqrt{r^2 - x^2}$ and $\sqrt{r^2 - y^2}$; and,

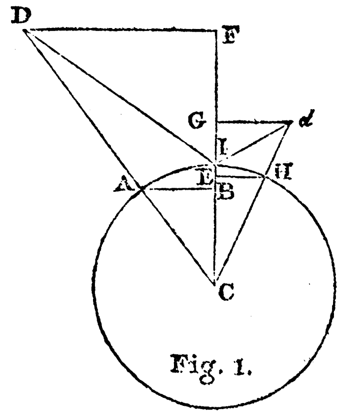

Fig. 1.

and, becaufe of the fimilar triangles ABC and BFC, HEC and dGC, $r : x :: a : \frac{ax}{r} =$ CE, $r : \sqrt{r^2 - x^2} :: a : \frac{a\sqrt{r^2 - x^2}}{r} =$ DF; $r : y :: b : \frac{by}{r} =$ CG, and $r : \sqrt{r^2 - y^2} :: b : \frac{b\sqrt{r^2 - y^2}}{r} = d$G; confequently, $\frac{ax}{r} - r =$ FI, $\frac{bx}{r} - r =$ GI; and becaufe the angles DIF, dIG, are equal by the nature of the problem, and the angles DFI and dGI both right angles, the triangles DFI and dGI are alfo fimilar, and confequently $\frac{a\sqrt{r^2 - x^2}}{r} : \frac{ax}{r} - r :: \frac{b\sqrt{r^2 - y^2}}{r} : \frac{by}{r} - r$; and $\frac{x}{\sqrt{r^2 - x^2}} - \frac{r^2}{a\sqrt{r^2 - x^2}} = \frac{y}{\sqrt{r^2 - y^2}} - \frac{r^2}{b\sqrt{r^2 - y^2}}$; or $\frac{r^2}{b\sqrt{r^2 - y^2}} - \frac{r^2}{a\sqrt{r^2 - x^2}} = \frac{y}{\sqrt{r^2 - y^2}} - \frac{x}{\sqrt{r^2 - x^2}}$: or, the co-fecant of the arc HI $\div\, b$ – co-fecant of AI $\div\, a =$ the co-tangent HI $\div\, r$ – co-tangent AI $\div\, r$; or, laftly, the co-tangent of HI – co-tangent of AI = co-fecant of HI $\times \frac{r}{b}$ – co-fecant AI $\times \frac{r}{a}$. Confequently, we have to find two arcs, the fum of which is given, and fuch that the difference of their co-tangents may be equal to the difference of the products of their co-fecants into given quantities.

To do this affume the angle DCF as near as poffible; and, becaufe the fum of the two angles is given, the angle dCG will be known alfo. Take the difference of the logarithms of r and a, r and b, which will be conftant, alfo the difference of the co-tangents of the two affumed arcs, and having taken out the log. co-fecants, add to them refpectively the two logarithmic differences.

differences. Find the numbers correfponding to thefe two fums, and if the difference of thefe two numbers be equal to the difference of the co-tangents, the angle DCF was rightly affumed; but as that will feldom happen, take the difference, or error; affume the angle DCF again, repeat the operation, and find the error as before. Then, as the fum of the errors, if one of them was too great, and the other too little, or their difference, if both were too great, or both too little, is to the difference of thefe affumptions, fo is the lefs error to a number of minutes and feconds, which muft be added to that affumption to which the leaft error belongs, if that affumption was too fmall; or fubtracted from it, if the affumption was too great: and, unlefs the firft affumption was made very wide of the truth, which may always be avoided, the two angles will generally be obtained within a few feconds of the truth, and, by repeating the operation once more, to the utmoft exactnefs.

Suppofe DC (a) be taken equal to 72, dC (b) = 48, and the radius CI (r) = 40, the angle DCd being 82° 45′: then the whole operation will ftand as follows:

$r = 40$ log.	11.6020600	- -	11.6020600
$a = 72$ log.	1.8573325	$b = 48$ log.	1.6812412
Conftant log.	9.7446275	Conftant log.	9.9208188

Now, in the two triangles DCI, dCI, the angles DIC and dIC being equal, and CI common, but dC confiderably lefs than DC, it is manifeft, that the angle dCI will be confiderably lefs than the angle DCI: let them be affumed in the proportion that DC bears to its excefs above dC; in which cafe the angle dCI will be 27° 35′ and DCI 55° 10′. The co-tangent of the former will be 1.9141795, of the latter .6958813; and the difference of them 1.2182982. The log. co-fecants of thofe

two

two angles are 10.3343832 and 10.0857536, which being respectively increased by 9.9208188 and 9.7446275, the two constant logarithms, make 0.2552020 and 9.8304811, which are the logarithms of 1.7997079 and .6768323; and the difference of these two numbers is 1.1228756, which is less than the difference of the log. co-tangents by .0954226.

I next assume the angles 30° and 52° 45′; and by pursuing the same steps which have been described above, I find the difference of their co-tangents exceeds the difference of the products by .0028987. Then, as 925239 (the difference of the errors) is to 145′ (the difference of suppositions), so is the latter error 28987 to 4′ 33″, which being added to 30°, gives 30° 4′ 33″ for the next assumption of the angle *d*CI; but for ease in the computation I shall take 30° 5′; in which case the angle DCI will be 52° 40′; and by repeating the operation the difference of the co-tangents will be found less than the difference of the products by .0002425. And 31412 (the sum of the two last errors) is to 5′ (the difference of the suppositions) as 2425 (the last error) is to 23″; which being taken from 30° 5′, the last supposition, because it was too great, leaves 30° 4′ 37″ for the exact value of the angle *d*CI.

This equation, like that in the fourth example, when the value of *y* is properly substituted, and the equation reduced in the usual manner, will rise to four dimensions with all the inferior ones; and it does not appear, that either HUYGENS, SLUSIUS, Mr. ROBINS, Dr. WILLSON, or Professor SIMSON, with all their artifice, have been able to depress it: but by this method of resolution the point of reflection is found, with the greatest exactness, in much less time than this substitution and reduction can be made. And this example farther suggests to us, that when the answer is sought by the method now under

2 consi-

consideration, it is not always necessary to exterminate all the unknown quantities but one.

EXAMPLE VII.

Suppose the equation to be resolved were $a = .375 = 16y^5 \mp 4y^4 - 20y^3 \pm 4y^2 + 5y$: and, first, let the upper signs have place, and it is manifest, that the latter side of the equation may be divided into two parts; namely, $4y^2 - 4y^4 = 4y^2 . \overline{1+y} . \overline{1-y}$, and $16y^5 - 20y^3 + 5y = y^5 - 10y^3 . \overline{1+y} . \overline{1-y} + 5y . \overline{1+y}|^2 . \overline{1-y}|^2$. But the former part is (by tab. I.) the square of the sine of twice the arc which has y for its sine (radius being $= 1$) and the latter part the sine of five times the same arc. Hence, therefore, the given quantity a ($= .375$) is equal to the sum of the sine of five times the arc (A) which has y for its sine, and the square of the sine of twice the same arc. Now, as the square of the sine of twice the arc (A) must necessarily in this instance be very small in comparison of the sine of five times the same arc (A), it is manifest, that the sine of five times the arc which has y for its sine will be very little less than .375, and of course that arc (5A) can be but very little less than 22° 2′, the sine of which is next greater than that number. Assume it 21°, and the fifth part of it, or that arc which has y for its sine, will be 4° 12′, the double of which is 8° 24′. Now the log. sine of 8° 24′ is 9.1645998, which being doubled is 8.3291996, the logarithm of .0213403, and this number being taken from .375 leaves .3536597, which ought to have been .3583679, the sine of 21°, and of course is too small by .0047082: the arc has, therefore, been assumed too great.

Let

Let 20° 45′ be next assumed; the fifth part of which is 4° 9′, and twice this last number is 8° 18′, of which the log. sine is 9.1594354; and this being doubled is 8.3188708, the log. of .0208387; and this being taken from .375 will leave .3541613: less still than the sine of 20° 45′ by .0001297.

Take now 20° 44′, the fifth of which is 4° 8′ 48″, and two-fifths is 8° 17′ 36″; and the log. sine of this is 9.1590889, which being doubled gives 8.3181778, the logarithm of .0208055; and this being taken from .375, leaves .3541945; more than the sine 20° 44′ by .0000795. Now 2092 (the sum of the last two last errors) is to 60″ as 795 (the last error) is to 23″. Which being added to 20° 44′, the last assumption, gives 20° 44′ 23″ for five times the arc of which y is the sine: y is therefore the sine of 4° 8′ 52″.6, or .07233202.

When the lower signs in the equation have place, the given quantity a will be equal to the excess of the sine of five times an arc above the square of the sine of twice that arc: and the operation, after assuming, from the circumstances of the question, or equation, an arc which is nearly five times that having y for its sine, is this. Find the logarithmic sine of two-fifths of that arc, double it, find the number corresponding to this logarithm, and add to it the value of a, which should then be equal to the sine of the arc first assumed; and if it is not, to repeat the operation until an error is obtained on each side, and not very distant from the truth, as is done above, and which may always be done with three assumptions.

A multitude of examples might be added from the writings of different authors, who have either left their conclusions

unexhibited in numbers, for want of fome fuch eafy method as this, or have done it by means of a long and laborious feries of difficult computations; which, befide the labour attending them, are always fubject to a variety of errors, which cannot be detected, in many cafes, without repeating the operation.